3 Multiple Integrals

3.1 Double Integrals

In single-variable calculus, differentiation and integration are thought of as inverse operations. For instance, to integrate a function $f(x)$ it is necessary to find the antiderivative of f, that is, another function $F(x)$ whose derivative is $f(x)$. Is there a similar way of defining integration of real-valued functions of two or more variables? The answer is yes, as we will see shortly. Recall also that the definite integral of a nonnegative function $f(x) \ge 0$ represented the area "under" the curve $y = f(x)$. As we will now see, the *double integral* of a nonnegative real-valued function $f(x,y) \ge 0$ represents the *volume* "under" the surface $z = f(x,y)$.

Let $f(x,y)$ be a continuous function such that $f(x,y) \ge 0$ for all (x,y) on the **rectangle** $R = \{(x,y) : a \le x \le b,\, c \le y \le d\}$ in $\mathbb{R}^2$. We will often write this as $R = [a,b] \times [c,d]$. For any number $x*$ in the interval $[a,b]$, slice the surface $z = f(x,y)$ with the plane $x = x*$ parallel to the yz-plane. Then the trace of the surface in that plane is the *curve* $f(x*,y)$, where $x*$ is fixed and only y varies. The area A under that curve (i.e. the area of the region between the curve and the xy-plane) as y varies over the interval $[c,d]$ then depends only on the value of $x*$. So using the variable x instead of $x*$, let $A(x)$ be that area (see Figure 3.1.1).

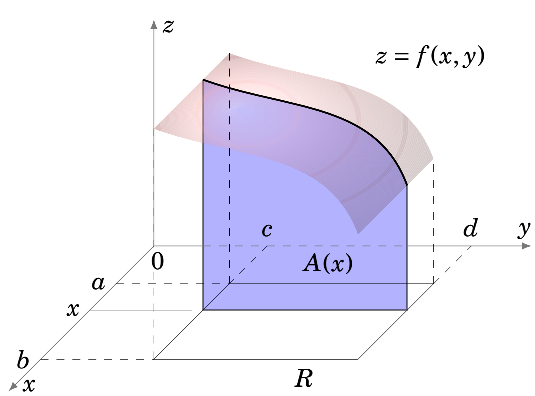


Figure 3.1.1 The area $A(x)$ varies with x

Then $A(x) = \int_c^d f(x,y)\,dy$ since we are treating x as fixed, and only y varies. This makes sense since for a fixed x the function $f(x,y)$ is a continuous function of y over the interval $[c,d]$, so we know that the area under the curve is the definite integral. The area $A(x)$ is a function of x, so by the "slice" or cross-section method from single-variable calculus we know that the volume V of the *solid* under the surface $z = f(x,y)$ but above the xy-plane over the

rectangle R is the integral over $[a,b]$ of that cross-sectional area $A(x)$:

$$V = \int_a^b A(x)\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx \tag{3.1}$$

We will always refer to this volume as "the volume under the surface". The above expression uses what are called **iterated integrals**. First the function $f(x,y)$ is integrated as a function of y, treating the variable x as a constant (this is called *integrating with respect to* y). That is what occurs in the "inner" integral between the square brackets in equation (3.1). This is the first iterated integral. Once that integration is performed, the result is then an expression involving only x, which can then be *integrated with respect to* x. That is what occurs in the "outer" integral above (the second iterated integral). The final result is then a number (the volume). This process of going through two iterations of integrals is called *double integration*, and the last expression in equation (3.1) is called a **double integral**.

Notice that integrating $f(x,y)$ with respect to y is the inverse operation of taking the partial derivative of $f(x,y)$ with respect to y. Also, we could just as easily have taken the area of cross-sections under the surface which were parallel to the xz-plane, which would then depend only on the variable y, so that the volume V would be

$$V = \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy\,. \tag{3.2}$$

It turns out that in general[1] the order of the iterated integrals does not matter. Also, we will usually discard the brackets and simply write

$$V = \int_c^d \int_a^b f(x,y)\,dx\,dy\,, \tag{3.3}$$

where it is understood that the fact that dx is written before dy means that the function $f(x,y)$ is first integrated with respect to x using the "inner" limits of integration a and b, and then the resulting function is integrated with respect to y using the "outer" limits of integration c and d. This order of integration can be changed if it is more convenient.

Example 3.1. Find the volume V under the plane $z = 8x + 6y$ over the rectangle $R = [0,1] \times [0,2]$.

[1] due to *Fubini's Theorem*. See Ch. 18 in TAYLOR and MANN.

Solution: We see that $f(x, y) = 8x + 6y \geq 0$ for $0 \leq x \leq 1$ and $0 \leq y \leq 2$, so:

$$\begin{aligned}
V &= \int_0^2 \int_0^1 (8x+6y)\,dx\,dy \\
&= \int_0^2 \left(4x^2 + 6xy \Big|_{x=0}^{x=1} \right) dy \\
&= \int_0^2 (4+6y)\,dy \\
&= 4y + 3y^2 \Big|_0^2 \\
&= 20
\end{aligned}$$

Suppose we had switched the order of integration. We can verify that we still get the same answer:

$$\begin{aligned}
V &= \int_0^1 \int_0^2 (8x+6y)\,dy\,dx \\
&= \int_0^1 \left(8xy + 3y^2 \Big|_{y=0}^{y=2} \right) dx \\
&= \int_0^1 (16x+12)\,dx \\
&= 8x^2 + 12x \Big|_0^1 \\
&= 20
\end{aligned}$$

Example 3.2. Find the volume V under the surface $z = e^{x+y}$ over the rectangle $R = [2,3] \times [1,2]$.

Solution: We know that $f(x, y) = e^{x+y} > 0$ for all (x, y), so

$$\begin{aligned}
V &= \int_1^2 \int_2^3 e^{x+y}\,dx\,dy \\
&= \int_1^2 \left(e^{x+y} \Big|_{x=2}^{x=3} \right) dy \\
&= \int_1^2 (e^{y+3} - e^{y+2})\,dy \\
&= e^{y+3} - e^{y+2} \Big|_1^2 \\
&= e^5 - e^4 - (e^4 - e^3) \;=\; e^5 - 2e^4 + e^3
\end{aligned}$$

Recall that for a general function $f(x)$, the integral $\int_a^b f(x)\,dx$ represents the difference of the area below the curve $y = f(x)$ but above the x-axis when $f(x) \geq 0$, and the area above the

curve but below the x-axis when $f(x) \le 0$. Similarly, the double integral of any continuous function $f(x, y)$ represents the difference of the volume below the surface $z = f(x, y)$ but above the xy-plane when $f(x, y) \ge 0$, and the volume above the surface but below the xy-plane when $f(x, y) \le 0$. Thus, our method of double integration by means of iterated integrals can be used to evaluate the double integral of *any* continuous function over a rectangle, regardless of whether $f(x, y) \ge 0$ or not.

Example 3.3. Evaluate $\int_0^{2\pi} \int_0^{\pi} \sin(x + y)\,dx\,dy$.

Solution: Note that $f(x, y) = \sin(x+y)$ is both positive and negative over the rectangle $[0,\pi] \times [0,2\pi]$. We can still evaluate the double integral:

$$\begin{aligned}
\int_0^{2\pi} \int_0^{\pi} \sin(x + y)\,dx\,dy &= \int_0^{2\pi} \left(-\cos(x + y)\Big|_{x=0}^{x=\pi} \right) dy \\
&= \int_0^{2\pi} (-\cos(y + \pi) + \cos y)\,dy \\
&= -\sin(y + \pi) + \sin y \Big|_0^{2\pi} \;=\; -\sin 3\pi + \sin 2\pi - (-\sin \pi + \sin 0) \\
&= 0
\end{aligned}$$

Exercises

A

For Exercises 1-4, find the volume under the surface $z = f(x, y)$ over the rectangle R.

1. $f(x, y) = 4xy$, $R = [0,1] \times [0,1]$

2. $f(x, y) = e^{x+y}$, $R = [0,1] \times [-1,1]$

3. $f(x, y) = x^3 + y^2$, $R = [0,1] \times [0,1]$

4. $f(x, y) = x^4 + xy + y^3$, $R = [1,2] \times [0,2]$

For Exercises 5-12, evaluate the given double integral.

5. $\int_0^1 \int_1^2 (1 - y)x^2\,dx\,dy$

6. $\int_0^1 \int_0^2 x(x + y)\,dx\,dy$

7. $\int_0^2 \int_0^1 (x + 2)\,dx\,dy$

8. $\int_{-1}^2 \int_{-1}^1 x(xy + \sin x)\,dx\,dy$

9. $\int_0^{\pi/2} \int_0^1 xy\cos(x^2 y)\,dx\,dy$

10. $\int_0^{\pi} \int_0^{\pi/2} \sin x \cos(y - \pi)\,dx\,dy$

11. $\int_0^2 \int_1^4 xy\,dx\,dy$

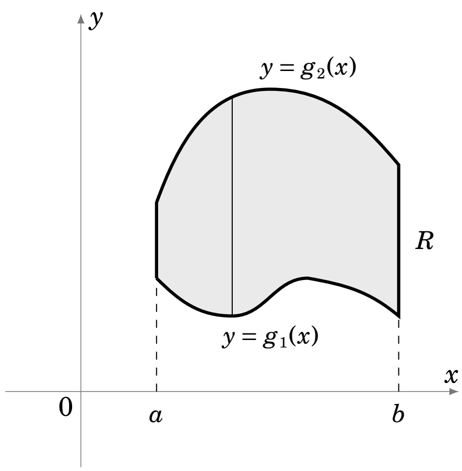
12. $\int_{-1}^1 \int_{-1}^2 1\,dx\,dy$

13. Let M be a constant. Show that $\int_c^d \int_a^b M\,dx\,dy = M(d - c)(b - a)$.

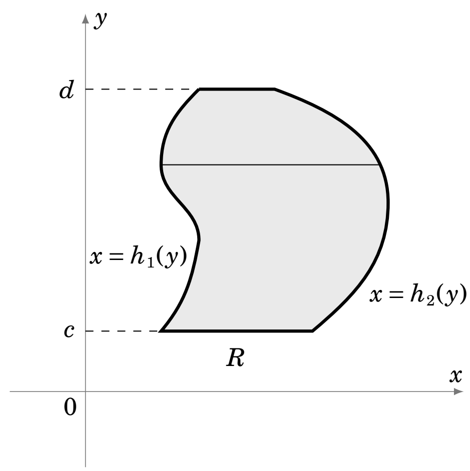
3.2 Double Integrals Over a General Region

In the previous section we got an idea of what a double integral over a rectangle represents. We can now define the double integral of a real-valued function $f(x,y)$ over more general regions in $\mathbb{R}^2$.

Suppose that we have a region R in the xy-plane that is bounded on the left by the vertical line $x = a$, bounded on the right by the vertical line $x = b$ (where $a < b$), bounded below by a curve $y = g_1(x)$, and bounded above by a curve $y = g_2(x)$, as in Figure 3.2.1(a). We will assume that $g_1(x)$ and $g_2(x)$ do not intersect on the open interval (a,b) (they could intersect at the endpoints $x = a$ and $x = b$, though).



(a) Vertical slice: $\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$



(b) Horizontal slice: $\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$

Figure 3.2.1 Double integral over a nonrectangular region R

Then using the slice method from the previous section, the double integral of a real-valued function $f(x,y)$ over the region R, denoted by $\iint\limits_R f(x,y)\,dA$, is given by

$$\iint\limits_R f(x,y)\,dA \;=\; \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right] dx \tag{3.4}$$

This means that we take vertical slices in the region R between the curves $y = g_1(x)$ and $y = g_2(x)$. The symbol dA is sometimes called an *area element* or *infinitesimal*, with the A signifying area. Note that $f(x,y)$ is first integrated with respect to y, with functions of x as the limits of integration. This makes sense since the result of the first iterated integral will have to be a function of x alone, which then allows us to take the second iterated integral with respect to x.

Similarly, if we have a region R in the xy-plane that is bounded on the left by a curve $x = h_1(y)$, bounded on the right by a curve $x = h_2(y)$, bounded below by the horizontal line

$y = c$, and bounded above by the horizontal line $y = d$ (where $c < d$), as in Figure 3.2.1(b) (assuming that $h_1(y)$ and $h_2(y)$ do not intersect on the open interval (c,d)), then taking horizontal slices gives

$$\iint_R f(x,y)\,dA \;=\; \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right] dy \tag{3.5}$$

Notice that these definitions include the case when the region R is a rectangle. Also, if $f(x,y) \ge 0$ for all (x,y) in the region R, then $\iint_R f(x,y)\,dA$ is the volume under the surface $z = f(x,y)$ over the region R.

Example 3.4. Find the volume V under the plane $z = 8x+6y$ over the region $R = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 2x^2\}$.

Solution: The region R is shown in Figure 3.2.2. Using vertical slices we get:

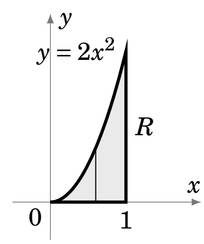


Figure 3.2.2

$$\begin{aligned} V &= \iint_R (8x+6y)\,dA \\ &= \int_0^1 \left[\int_0^{2x^2} (8x+6y)\,dy\right] dx \\ &= \int_0^1 \left(8xy + 3y^2 \Big|_{y=0}^{y=2x^2}\right) dx \\ &= \int_0^1 (16x^3 + 12x^4)\,dx \\ &= 4x^4 + \tfrac{12}{5}x^5 \Big|_0^1 \;=\; 4 + \tfrac{12}{5} \;=\; \tfrac{32}{5} \;=\; 6.4 \end{aligned}$$

We get the same answer using horizontal slices (see Figure 3.2.3):

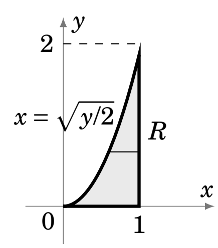


Figure 3.2.3

$$\begin{aligned} V &= \iint_R (8x+6y)\,dA \\ &= \int_0^2 \left[\int_{\sqrt{y/2}}^1 (8x+6y)\,dx\right] dy \\ &= \int_0^2 \left(4x^2 + 6xy \Big|_{x=\sqrt{y/2}}^{x=1}\right) dy \\ &= \int_0^2 (4+6y-(2y+\tfrac{6}{\sqrt{2}}y\sqrt{y}))\,dy \;=\; \int_0^2 (4+4y-3\sqrt{2}y^{3/2})\,dy \\ &= 4y+2y^2-\tfrac{6\sqrt{2}}{5}y^{5/2}\Big|_0^2 \;=\; 8+8-\tfrac{6\sqrt{2}\sqrt{32}}{5} \;=\; 16-\tfrac{48}{5} \;=\; \tfrac{32}{5} \;=\; 6.4 \end{aligned}$$

Example 3.5. Find the volume V of the solid bounded by the three coordinate planes and the plane $2x+y+4z=4$.

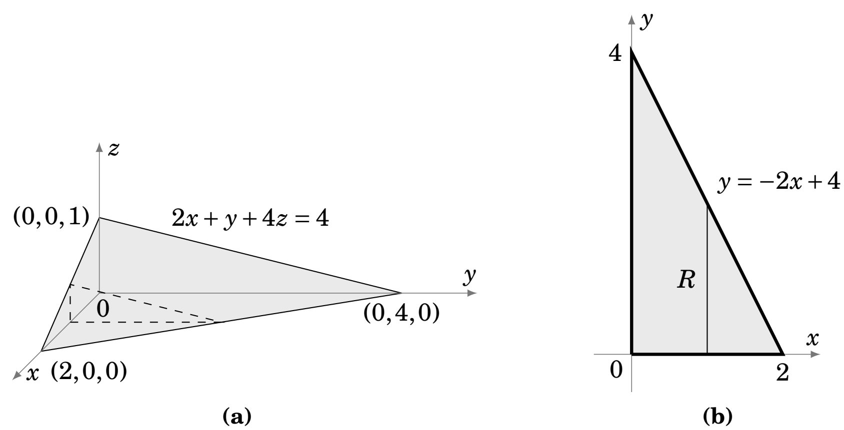


Figure 3.2.4

Solution: The solid is shown in Figure 3.2.4(a) with a typical vertical slice. The volume V is given by $\iint\limits_R f(x,y)\,dA$, where $f(x,y)=z=\frac{1}{4}(4-2x-y)$ and the region R, shown in Figure 3.2.4(b), is $R=\{(x,y):0\le x\le 2,\ 0\le y\le -2x+4\}$. Using vertical slices in R gives

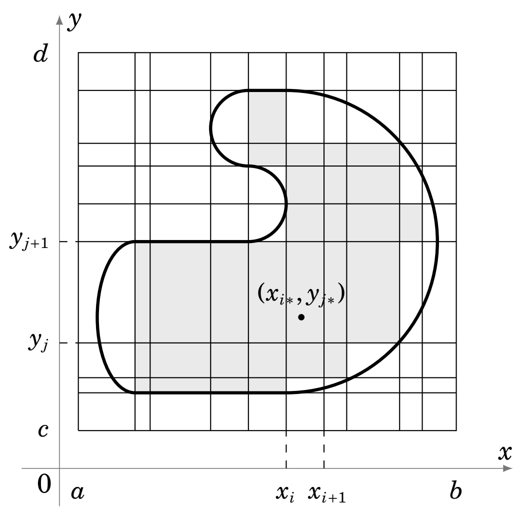
$$\begin{aligned}
V &= \iint\limits_R \tfrac{1}{4}(4-2x-y)\,dA \\
&= \int_0^2 \left[\int_0^{-2x+4} \tfrac{1}{4}(4-2x-y)\,dy\right] dx \\
&= \int_0^2 \left(-\tfrac{1}{8}(4-2x-y)^2 \Big|_{y=0}^{y=-2x+4}\right) dx \\
&= \int_0^2 \tfrac{1}{8}(4-2x)^2\,dx \\
&= -\tfrac{1}{48}(4-2x)^3 \Big|_0^2 \;=\; \tfrac{64}{48} \;=\; \tfrac{4}{3}
\end{aligned}$$

For a general region R, which may not be one of the types of regions we have considered so far, the double integral $\iint\limits_R f(x,y)\,dA$ is defined as follows. Assume that $f(x,y)$ is a nonnegative real-valued function and that R is a bounded region in $\mathbb{R}^2$, so it can be enclosed in some rectangle $[a,b]\times[c,d]$. Then divide that rectangle into a grid of subrectangles. Only consider the subrectangles that are enclosed completely within the region R, as shown by the shaded subrectangles in Figure 3.2.5(a). In any such subrectangle $[x_i,x_{i+1}]\times[y_j,y_{j+1}]$, pick a point (x_{i*},y_{j*}). Then the volume under the surface $z=f(x,y)$ over that subrectangle is approximately $f(x_{i*},y_{j*})\,\Delta x_i\,\Delta y_j$, where $\Delta x_i=x_{i+1}-x_i$, $\Delta y_j=y_{j+1}-y_j$, and $f(x_{i*},y_{j*})$ is the height and

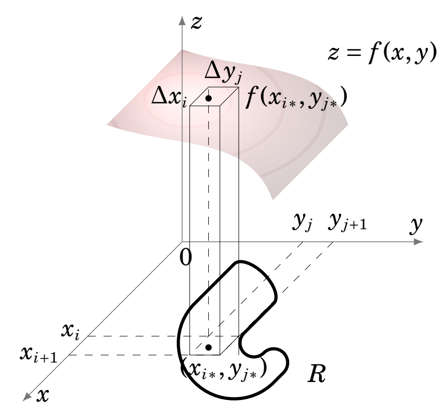
$\Delta x_i\,\Delta y_j$ is the base area of a parallelepiped, as shown in Figure 3.2.5(b). Then the total volume under the surface is approximately the sum of the volumes of all such parallelepipeds, namely

$$\sum_j \sum_i f(x_{i*}, y_{j*})\,\Delta x_i\,\Delta y_j\ , \tag{3.6}$$

where the summation occurs over the indices of the subrectangles inside R. If we take smaller and smaller subrectangles, so that the length of the largest diagonal of the subrectangles goes to 0, then the subrectangles begin to fill more and more of the region R, and so the above sum approaches the actual volume under the surface $z = f(x,y)$ over the region R. We then *define* $\iint\limits_R f(x,y)\,dA$ as the limit of that double summation (the limit is taken over all subdivisions of the rectangle $[a,b]\times[c,d]$ as the largest diagonal of the subrectangles goes to 0).



(a) Subrectangles inside the region R



(b) Parallelepiped over a subrectangle, with volume $f(x_{i*}, y_{j*})\,\Delta x_i\,\Delta y_j$

Figure 3.2.5 Double integral over a general region R

A similar definition can be made for a function $f(x,y)$ that is not necessarily always non-negative: just replace each mention of volume by the negative volume in the description above when $f(x,y) < 0$. In the case of a region of the type shown in Figure 3.2.1, using the definition of the Riemann integral from single-variable calculus, our definition of $\iint\limits_R f(x,y)\,dA$ reduces to a sequence of two iterated integrals.

Finally, the region R does not have to be bounded. We can evaluate *improper* double integrals (i.e. over an unbounded region, or over a region which contains points where the function $f(x,y)$ is not defined) as a sequence of iterated improper single-variable integrals.

Example 3.6. Evaluate $\int_1^\infty \int_0^{1/x^2} 2y\,dy\,dx$.

Solution:

$$\begin{aligned}\int_1^\infty \int_0^{1/x^2} 2y\,dy\,dx &= \int_1^\infty \left(y^2 \Big|_{y=0}^{y=1/x^2} \right) dx \\ &= \int_1^\infty x^{-4}\,dx \;=\; -\tfrac{1}{3}x^{-3}\Big|_1^\infty \;=\; 0-(-\tfrac{1}{3}) \;=\; \tfrac{1}{3}\end{aligned}$$

Exercises

A

For Exercises 1-6, evaluate the given double integral.

1. $\int_0^1 \int_{\sqrt{x}}^1 24x^2 y\,dy\,dx$

2. $\int_0^\pi \int_0^y \sin x\,dx\,dy$

3. $\int_1^2 \int_0^{\ln x} 4x\,dy\,dx$

4. $\int_0^2 \int_0^{2y} e^{y^2}\,dx\,dy$

5. $\int_0^{\pi/2} \int_0^y \cos x \sin y\,dx\,dy$

6. $\int_0^\infty \int_0^\infty xye^{-(x^2+y^2)}\,dx\,dy$

7. $\int_0^2 \int_0^y 1\,dx\,dy$

8. $\int_0^1 \int_0^{x^2} 2\,dy\,dx$

9. Find the volume V of the solid bounded by the three coordinate planes and the plane $x+y+z=1$.

10. Find the volume V of the solid bounded by the three coordinate planes and the plane $3x+2y+5z=6$.

B

11. Explain why the double integral $\iint\limits_R 1\,dA$ gives the area of the region R. For simplicity, you can assume that R is a region of the type shown in Figure 3.2.1(a).

C

12. Prove that the volume of a tetrahedron with mutually perpendicular adjacent sides of lengths a, b, and c, as in Figure 3.2.6, is $\frac{abc}{6}$. (*Hint: Mimic Example 3.5, and recall from Section 1.5 how three noncollinear points determine a plane.*)

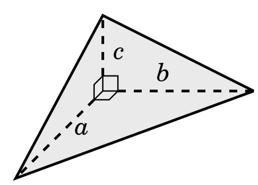


Figure 3.2.6

13. Show how Exercise 12 can be used to solve Exercise 10.

3.3 Triple Integrals

Our definition of a double integral of a real-valued function $f(x,y)$ over a region R in $\mathbb{R}^2$ can be extended to define a *triple integral* of a real-valued function $f(x,y,z)$ over a *solid* S in $\mathbb{R}^3$. We simply proceed as before: the solid S can be enclosed in some rectangular parallelepiped, which is then divided into subparallelepipeds. In each subparallelepiped inside S, with sides of lengths Δx, Δy and Δz, pick a point (x_*, y_*, z_*). Then define the triple integral of $f(x,y,z)$ over S, denoted by $\iiint\limits_S f(x,y,z)\,dV$, by

$$\iiint\limits_S f(x,y,z)\,dV \;=\; \lim \sum\sum\sum f(x_*, y_*, z_*)\,\Delta x\,\Delta y\,\Delta z\;, \tag{3.7}$$

where the limit is over all divisions of the rectangular parallelepiped enclosing S into subparallelepipeds whose largest diagonal is going to 0, and the triple summation is over all the subparallelepipeds inside S. It can be shown that this limit does not depend on the choice of the rectangular parallelepiped enclosing S. The symbol dV is often called the *volume element*.

Physically, what does the triple integral represent? We saw that a double integral could be thought of as the volume under a two-dimensional surface. It turns out that the triple integral simply generalizes this idea: it can be thought of as representing the *hypervolume* under a three-dimensional *hypersurface* $w = f(x,y,z)$ whose graph lies in $\mathbb{R}^4$. In general, the word "volume" is often used as a general term to signify the same concept for any n-dimensional object (e.g. length in $\mathbb{R}^1$, area in $\mathbb{R}^2$). It may be hard to get a grasp on the concept of the "volume" of a four-dimensional object, but at least we now know how to calculate that volume!

In the case where S is a rectangular parallelepiped $[x_1,x_2]\times[y_1,y_2]\times[z_1,z_2]$, that is, $S = \{(x,y,z) : x_1 \le x \le x_2,\; y_1 \le y \le y_2,\; z_1 \le z \le z_2\}$, the triple integral is a sequence of three iterated integrals, namely

$$\iiint\limits_S f(x,y,z)\,dV \;=\; \int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y,z)\,dx\,dy\,dz\;, \tag{3.8}$$

where the order of integration does not matter. This is the simplest case.

A more complicated case is where S is a solid which is bounded below by a surface $z = g_1(x,y)$, bounded above by a surface $z = g_2(x,y)$, y is bounded between two curves $h_1(x)$ and $h_2(x)$, and x varies between a and b. Then

$$\iiint\limits_S f(x,y,z)\,dV \;=\; \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz\,dy\,dx\;. \tag{3.9}$$

Notice in this case that the first iterated integral will result in a function of x and y (since its limits of integration are functions of x and y), which then leaves you with a double integral of

a type that we learned how to evaluate in Section 3.2. There are, of course, many variations on this case (for example, changing the roles of the variables x, y, z), so as you can probably tell, triple integrals can be quite tricky. At this point, just learning how to evaluate a triple integral, regardless of what it represents, is the most important thing. We will see some other ways in which triple integrals are used later in the text.

Example 3.7. Evaluate $\int_0^3 \int_0^2 \int_0^1 (xy+z)\,dx\,dy\,dz$.

Solution:

$$\begin{aligned}
\int_0^3 \int_0^2 \int_0^1 (xy+z)\,dx\,dy\,dz &= \int_0^3 \int_0^2 \left(\tfrac{1}{2}x^2 y + xz \Big|_{x=0}^{x=1} \right) dy\,dz \\
&= \int_0^3 \int_0^2 \left(\tfrac{1}{2}y + z \right) dy\,dz \\
&= \int_0^3 \left(\tfrac{1}{4}y^2 + yz \Big|_{y=0}^{y=2} \right) dz \\
&= \int_0^3 (1+2z)\,dz \\
&= z + z^2 \Big|_0^3 \;=\; 12
\end{aligned}$$

Example 3.8. Evaluate $\int_0^1 \int_0^{1-x} \int_0^{2-x-y} (x+y+z)\,dz\,dy\,dx$.

Solution:

$$\begin{aligned}
\int_0^1 \int_0^{1-x} \int_0^{2-x-y} (x+y+z)\,dz\,dy\,dx &= \int_0^1 \int_0^{1-x} \left((x+y)z + \tfrac{1}{2}z^2 \Big|_{z=0}^{z=2-x-y} \right) dy\,dx \\
&= \int_0^1 \int_0^{1-x} \left((x+y)(2-x-y) + \tfrac{1}{2}(2-x-y)^2 \right) dy\,dx \\
&= \int_0^1 \int_0^{1-x} \left(2 - \tfrac{1}{2}x^2 - xy - \tfrac{1}{2}y^2 \right) dy\,dx \\
&= \int_0^1 \left(2y - \tfrac{1}{2}x^2 y - xy - \tfrac{1}{2}xy^2 - \tfrac{1}{6}y^3 \Big|_{y=0}^{y=1-x} \right) dx \\
&= \int_0^1 \left(\tfrac{11}{6} - 2x + \tfrac{1}{6}x^3 \right) dx \\
&= \tfrac{11}{6}x - x^2 + \tfrac{1}{24}x^4 \Big|_0^1 \;=\; \tfrac{7}{8}
\end{aligned}$$

Note that the volume V of a solid in $\mathbb{R}^3$ is given by

$$V = \iiint_S 1\,dV\,. \tag{3.10}$$

Since the function being integrated is the constant 1, then the above triple integral reduces to a double integral of the types that we considered in the previous section if the solid is bounded above by some surface $z = f(x,y)$ and bounded below by the xy-plane $z = 0$. There are many other possibilities. For example, the solid could be bounded below and above by surfaces $z = g_1(x,y)$ and $z = g_2(x,y)$, respectively, with y bounded between two curves $h_1(x)$ and $h_2(x)$, and x varies between a and b. Then

$$V = \iiint_S 1\,dV = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} 1\,dz\,dy\,dx = \int_a^b \int_{h_1(x)}^{h_2(x)} (g_2(x,y) - g_1(x,y))\,dy\,dx$$

just like in equation (3.9). See Exercise 10 for an example.

Exercises

A

For Exercises 1-8, evaluate the given triple integral.

1. $\displaystyle\int_0^3 \int_0^2 \int_0^1 xyz\,dx\,dy\,dz$

2. $\displaystyle\int_0^1 \int_0^x \int_0^y xyz\,dz\,dy\,dx$

3. $\displaystyle\int_0^\pi \int_0^x \int_0^{xy} x^2 \sin z\,dz\,dy\,dx$

4. $\displaystyle\int_0^1 \int_0^z \int_0^y z e^{y^2}\,dx\,dy\,dz$

5. $\displaystyle\int_1^e \int_0^y \int_0^{1/y} x^2 z\,dx\,dz\,dy$

6. $\displaystyle\int_1^2 \int_0^{y^2} \int_0^{z^2} yz\,dx\,dz\,dy$

7. $\displaystyle\int_1^2 \int_2^4 \int_0^3 1\,dx\,dy\,dz$

8. $\displaystyle\int_0^1 \int_0^{1-x} \int_0^{1-x-y} 1\,dz\,dy\,dx$

9. Let M be a constant. Show that $\int_{z_1}^{z_2} \int_{y_1}^{y_2} \int_{x_1}^{x_2} M\,dx\,dy\,dz = M(z_2 - z_1)(y_2 - y_1)(x_2 - x_1)$.

B

10. Find the volume V of the solid S bounded by the three coordinate planes, bounded above by the plane $x + y + z = 2$, and bounded below by the plane $z = x + y$.

C

11. Show that $\displaystyle\int_a^b \int_a^z \int_a^y f(x)\,dx\,dy\,dz = \int_a^b \frac{(b-x)^2}{2} f(x)\,dx$. (*Hint: Think of how changing the order of integration in the triple integral changes the limits of integration.*)